

# ON THE COUNTABLE, MEASURE PRESERVING RELATION INDUCED AN HOMOGENEOUS QUOTIENT, BY THE ACTION OF A DISCRETE GROUP

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**ABSTRACT.** In this paper we consider a countable discrete group  $G$  acting ergodically by measure preserving transformation on an infinite measure space  $(\mathcal{X}, \nu)$ , with  $\sigma$ -finite measure  $\nu$ . In addition we assume that  $\Gamma \subseteq G$  is an almost normal subgroup, that has a fundamental domain  $F$  of finite measure in  $\mathcal{X}$ . We consider the countable measurable equivalence relation  $\mathcal{R}_G$  on  $\mathcal{X}$  induced by the orbits of  $G$ , and let  $\mathcal{R}_G|_F$  be its restriction to  $F$  (thus two points in  $F$  are equivalent if and only if they are on the same orbit of  $G$ ). The  $C^*$ -algebra groupoid structure corresponding to such a quotient was studied in [LLN].

In this paper we analyse the generators and relations for this algebra in the case  $G = \mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}])$ ,  $\Gamma = \mathrm{PSL}_2(\mathbb{R})$ .

Let  $G$  be a countable, discrete group acting ergodically, by measure preserving transformation on an infinite measure space  $(\mathcal{X}, \nu)$ , with  $\sigma$ -finite measure  $\nu$ . In addition we assume that  $\Gamma \subseteq G$  is an almost normal subgroup, that has a fundamental domain  $F$  of finite measure in  $\mathcal{X}$ . We consider the countable measurable equivalence relation  $\mathcal{R}_G$  on  $\mathcal{X}$  induced by the orbits of  $G$ , and let  $\mathcal{R}_G|_F$  be its restriction to  $F$  (thus two points in  $F$  are equivalent if and only if they are on the same orbit of  $G$ ). The  $C^*$ -algebra groupoid structure corresponding to such a quotient was studied in [LLN].

In this paper we analyse the generators and relations for this algebra in the case  $G = \mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}])$ ,  $\Gamma = \mathrm{PSL}_2(\mathbb{R})$  and determine the precise composition relations for the generators for the  $*$ -algebra associated to the equivalence relation  $\mathcal{R}_G|_F$ . This will allow us to prove that for  $G = \mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}])$ ,  $p \geq 3$

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a prime,  $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ , if the action is a.e. free, then  $\mathcal{R}_G|F$  is treeable and has cost  $\frac{p+1}{2}$ . Hence, by the results of Hjorth, the equivalence relation is implemented by the action of the free group with  $\frac{p+1}{2}$  generators on  $F$ . Moreover, the radial algebra of the free group (the algebra generated by convolutors in the words on  $F_{\frac{p+1}{2}}$  of equal length have equal weight) will coincide with the Hecke algebra corresponding to  $G, \Gamma$  and to the action on  $\mathcal{X}$ .

Moreover, we give an explicit description for the action of the generators of  $\mathcal{R}_G|F$  on  $\mathcal{X}$ , which is context free (does not depend on  $\mathcal{X}$ ) in the sense of symbolic dynamics.

In particular, we prove that in analogy with the measured equivalence for groups ([Ga]), that  $\mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}])$  is infinitesimally orbit equivalent to  $F_{\frac{p+1}{2}}$ ,  $p \geq 3$  (see Corollary 8 for the definition of infinitesimally orbit equivalence).

We start with the construction of a family of generators for the relation  $\mathcal{R}_G|F$ .

**Proposition 1.** *Let  $G$  be a discrete group acting by measure preserving transformations, almost everywhere free  $\mathcal{X}$ . Assume that  $\Gamma$  is an almost normal subgroup, having a fundamental domain  $F \subseteq \mathcal{X}$ , of finite measure.*

*Let as above  $\mathcal{R}_G|F$  be the countable equivalence relation on  $F$ , defined by requiring that  $x \sim y$  if  $Gx = Gy$ .*

*For  $g$  in  $G$ , define the transformation  $\hat{\Gamma}g$  on  $F$  as follows:*

*Let  $x$  be in  $F$ , since  $F$  is a fundamental domain, there exists a unique  $\gamma \in \Gamma$  and  $x_1$  in  $F$  such that  $gx = \gamma_1 x_1$ .*

*We define  $\hat{\Gamma}gx = x_1 = \gamma_1^{-1}gx$ . Clearly,  $\hat{\Gamma}g$  depends only on the left  $\Gamma$ -coset class of  $g$ .*

*Then  $\mathcal{R}_G|F$  is generated by the transformations  $\hat{\Gamma}g$ ,  $g$  running through a system of representatives for right cosets of  $\Gamma$  (in the sense that  $x \sim y$  iff and only if there exists  $g \in G$  such that  $\hat{\Gamma}gx = y$ ).*

*Moreover,  $\hat{\Gamma}g$  is not injective, but the number of preimages of each point in the image is bounded by  $[\Gamma : \Gamma_g]$ , where  $\Gamma_g$  is the subgroup  $\Gamma \cap g\Gamma g^{-1}$ . In addition, if  $\Gamma g s_i$  are the left  $\Gamma$ -coset of  $\Gamma g \Gamma$ , then every point  $x$  in  $F$  will show up exactly  $[\Gamma : \Gamma_g]$ -times in the reunion of the images of the maps  $\hat{\Gamma}g s_i$ . Moreover, the same is true for preimages (with  $[\Gamma : \Gamma_{g^{-1}}]$  instead of  $[\Gamma : \Gamma_g]$ ).*

Note that with the above notations one could define a two cocycle  $\alpha = (\alpha_1, \alpha_2) : G \times F$  with values in  $\Gamma \times F$ , by defining  $\alpha(g, x) = (\gamma_1, x_1)$ . This cocycle could then be used to construct the adelic action of  $G$ , which would be implemented by the same cocycle.

*Proof.* The only thing to prove here is the statement about counting images and preimages. But this follows from the fact that the domain  $F_0 = \bigcup g s_i F$  is fundamental domain for  $\Gamma_{g^{-1}}$  and it covers  $[\Gamma : \Gamma_g]$  times the set  $F$ . Here  $s_i$  are coset representatives for  $\gamma g$  into  $\Gamma$ .  $\square$

The transformations  $\widehat{\Gamma g}$ ,  $g \in G$  have a natural composition rule. The composition rules are similar to the multiplication rules from the Hecke algebra, just that they have to be taken on pieces.

**Proposition 2.** *With the previous hypothesis, let  $g_1, g_2$  be arbitrary elements in  $G$ . Assume that  $r_j$ ,  $j = 1, 2, \dots, [\Gamma : \Gamma_{g_1^{-1}}]$  are the right coset representatives for  $\Gamma_{g_1^{-1}}$  in  $\Gamma$ . Thus  $\Gamma = \bigcup \Gamma_{g_1^{-1}} r_j$ , and hence  $\Gamma g_1 \Gamma = \bigcup \Gamma g_1 r_j$ .*

*Let  $A_{g_1, g_2}^{r_j}$  be the subset of  $F$  defined as*

$$\{f \in F \mid r_j g_2 f \in \Gamma_{g_1^{-1}} F\} = (r_j g_2)^{-1} \Gamma_{g_1^{-1}} F \cap F, \quad j = 1, 2, \dots, [\Gamma : \Gamma_{g_1^{-1}}].$$

*Let  $\chi_{A_{g_1, g_2}^{r_j}}$  be the characteristic functions of these sets.*

*Then*

$$\widehat{\Gamma g_1} \widehat{\Gamma g_2} = \sum_j \widehat{\Gamma g_1 r_j g_2} \chi_{A_{g_1, g_2}^{r_j}}.$$

*Moreover, the sets  $A_{g_1, g_2}^{r_j}$ ,  $j = 1, 2, \dots, [\Gamma : \Gamma_{g_1^{-1}}]$  are a partition of unity of  $F$ .*

*Proof.* Since  $G$  acts freely almost everywhere, we may simply work on are orbit of  $G$ . So we way assume that  $X = G$ , and that  $F = S$  is a system of coset representatives for  $\Gamma \setminus G$ . (The only point where the initial data would enter would be in the measure of the sets in the partitions from the previous proposition.)

Given  $s \in S$  and two left cosets  $\Gamma g_1, \Gamma g_2$  we calculate the composition  $\widehat{\Gamma g_1} \widehat{\Gamma g_2 s}$ .

Thus assume that  $g_2 s = \gamma_2 s_2$  for some  $\gamma_2 \in \Gamma$ ,  $s_2 \in S$  and thus  $\widehat{\Gamma g_2 s} = s_2$ .

Then  $s_2 = \gamma_2^{-1} g_2 s$  and hence

$$g_1(\widehat{\Gamma g_2 s}) = g_1 s_2 = g_1 \gamma_2^{-1} g_2 s.$$

We need to identify to which coset of  $\Gamma_{g_1^{-1}}$  the element  $\gamma_2^{-1}$  belongs. Assume thus that  $\gamma_2^{-1}$  belongs to  $\Gamma_{g_1^{-1}} r_j$  for some fixed  $j$ , thus  $\gamma_2^{-1} = \theta r_j$ ,  $\theta \in \Gamma_{g_1^{-1}}$ .

Then  $g_1\gamma_2^{-1}g_2s$  is further equal to  $(\gamma_1\theta\gamma_1^{-1})g_1r_jg_2s$ . But  $\theta' = \gamma_1\theta\gamma_1^{-1}$  belongs to  $\Gamma_g \subseteq \Gamma$  (since  $g\Gamma_{g^{-1}}g = g(\Gamma \cap g^{-1}\Gamma g)g = |\Gamma_g|$ ). Thus

$$g_1(\widehat{\Gamma g_2 s}) = g_1\gamma_2^{-1}g_2s = \theta'(g_1r_jg_2)s.$$

On the other hand, there exists  $\gamma_1 \in \Gamma$  such that  $g_1r_jg_2s = \gamma_1s_1$ ,  $s_1 \in s$ . Thus  $\widehat{\Gamma g_1r_jg_2s} = s_1$ . From the above formula we conclude

$$g_1(\widehat{\Gamma g_2 s}) = \theta'\gamma_1s_1$$

and hence

$$(*) \quad \widehat{\Gamma g_1} \widehat{\Gamma g_2} s = s_1 = \widehat{\Gamma g_1r_jg_2s}.$$

We have to determine the conditions that we have to impose  $s$ , so that  $\gamma_2$  belongs to  $\Gamma_{g_1^{-1}r_j}$ . But the relation defining  $s_2$  was

$$g_2s = \gamma_2s_2.$$

Thus for  $\gamma_2^{-1}$  to be in  $\Gamma_{g_1^{-1}r_j}$ , which is equivalent to  $\gamma_2 \in r_j^{-1}\Gamma_{g_1^{-1}}$ , is necessary and sufficient that  $g_2s$  belongs to  $r_j^{-1}\Gamma_{g_1^{-1}}S$ .

Thus  $s$  should belong to  $A_{g_1, g_2}^{r_j} = g_2^{-1}r_j^{-1}\Gamma_{g_1^{-1}}S \cap S$ . Thus the relation  $*$  holds on  $A_{g_1, g_2}^{r_j}$ .

Since the cosets  $r_j^{-1}\Gamma_{g_1^{-1}}$  are disjoint and  $S$  is a set representatives, it follows that  $\gamma S \cap S = \emptyset$  for all  $\gamma \neq e$  and hence  $\gamma_1S \cap \gamma_2S = \emptyset$  if  $\gamma_1 \neq \gamma_2$  and hence that  $r_j^{-1}g_2^{-1}s \cup r_k|_{g_1^{-1}}s = \psi'$ , if  $j \neq k$ . From here it follows that the sets  $A_{g_1, g_2}^{r_j}$ ,  $j = 1, 2, \dots, [\Gamma : \Gamma_{g_1^{-1}}]$  are forming a partition of  $S$  (since  $\bigcup A_{g_1, g_2}^{r_j} = g\Gamma S \cap S = gGS \cap S = G \cap S = S$ ).

Note that obviously the decomposition

$$\widehat{\Gamma g_1} \cdot \widehat{\Gamma g_2} = \sum_j \widehat{\Gamma g_1r_jg_2} \chi_{g_2^{-1}r_j^{-1}\Gamma_{g_1^{-1}}S \cap S}$$

depends only on the class  $\Gamma g_1$  of  $g_1$  (as  $\Gamma_{g_1^{-1}} = g_1^{-1}\Gamma g_1 \cap \Gamma$ ).

The formula does not depend either of the choice the representative  $g_2$  in  $\Gamma g_2$ , since changing  $g_2$  into  $\gamma'g_2$  would have the effect of permuting the sum, since

$$\Gamma_{g_1^{-1}}r_j\gamma' = \Gamma_{g_1^{-1}}r_{\pi_{\gamma'}(j)}$$

for some partition  $\pi_{\gamma'}$  of  $\{1, 2, \dots, [\Gamma : \Gamma_{g_1^{-1}}]\}$ . By using the methods from the previous proof, we prove the following.

**Lemma 3.** *Let  $S$  be as in the proof of the previous lemma. Let  $g \in G$  and let  $\alpha_i$  be a system of right representatives for cosets of  $\Gamma_g$  in  $\Gamma$  (that is  $\Gamma = \bigcup \Gamma_g \alpha_i$  or  $\Gamma g \Gamma = \bigcup \Gamma g \alpha_i$ ). Then for every  $\alpha_i$  the image through  $\widehat{\Gamma g}$  of the set  $g^{-1} \Gamma_g \alpha_i S \cap S = \{s \in S \mid gs \in \Gamma_g \alpha_i S\} = A_{\alpha_i, \Gamma g}$  is  $\alpha_i^{-1} \Gamma_g g S \cap S = B_{\alpha_i, g}$ .*

*Note that as before the sets  $A_{\alpha_i, \Gamma g}$  are partition of  $S$ , while the sets  $B_{\alpha_i, g}$  are not a partition in general; they may have overlaps  $S$ .*

*Moreover,  $\widehat{\Gamma g}|_{A_{\alpha_i, \Gamma g}}$  is bijective and the inverse is  $\widehat{\Gamma g^{-1} \alpha_i}$ , acting on  $\alpha_i^{-1} \Gamma_g g S \cap S$ .*

*Proof.* The fact that the image through  $\widehat{\Gamma g}$  of the set  $A_{\alpha_i, \Gamma g}$  is  $\alpha_i^{-1} \Gamma_g g S \cap S$ , is proved as follows.

Let  $s$  be an element in

$$A_{\alpha_i, \Gamma g} = g^{-1} \Gamma_g \alpha_i^{-1} S \cap S = \{s \in S \mid gs \in \Gamma_g \alpha_i S\}.$$

Thus  $gs = \theta \alpha_i s_1$  for some  $s_1 \in S$ ,  $\theta \in \Gamma_g$ . but then  $s_1 = \alpha_i^{-1} \theta g s$  which belongs to  $\alpha_i^{-1} \Gamma_g g S \cap S$ .

To verify the inverse formula we have to calculate  $\widehat{\Gamma g^{-1} \alpha_i} \widehat{\Gamma g}$ . By the previous proposition we have to choose  $r_j$ , a system of right representatives for  $\Gamma_{(g^{-1} \alpha_i)^{-1}}$  in  $\Gamma$ , that is  $\Gamma = \bigcup_i \Gamma_{(g^{-1} \alpha_i)^{-1}} r_j$ . But  $\Gamma_{(g^{-1} \alpha_i)^{-1}} = \Gamma_{\alpha_i^{-1} g} = \alpha_i^{-1} \Gamma_g \alpha_i$ , thus  $\Gamma = \bigcup (\alpha_i^{-1} \Gamma_g \alpha_i) r_j$ .

Then the previous formula gives that

$$\widehat{\Gamma g^{-1} \alpha_i} \widehat{\Gamma g} = \sum_j \widehat{\Gamma g^{-1} \alpha_i r_j g} \chi_{(g^{-1} r_j^{-1} \Gamma_{\alpha_i^{-1} g} S \cap S)}.$$

In the above formula, we get the identity exactly when  $\alpha_i r_j$  belongs to  $\Gamma_g$ . Then the identity term will occur on the set  $g^{-1} r_j^{-1} \Gamma_{\alpha_i^{-1} g} S \cap S = g^{-1} r_j^{-1} \alpha_i^{-1} \Gamma_g \alpha_i S \cap S$ , when  $\alpha_i r_j$  belongs to  $\Gamma_g$ . But in this case the set is  $g(\alpha_i r_j)^{-1} \Gamma_g \alpha_i S \cap S = g^{-1} \Gamma_g \alpha_i S \cap S$ . Thus the inverse of  $\widehat{\Gamma g}$  on  $g^{-1} \Gamma_g \alpha_i S \cap S$  is  $\widehat{\Gamma g^{-1} \alpha_i}$ .

It is easy to see that this formula is consistent, that is if we apply formula this to  $\widehat{\Gamma g^{-1} \alpha_i}$  on  $\alpha_i \Gamma_g g S \cap S$  we get the same result.

**Observation 4.** In general if we want to compute the inverses of all  $\widehat{\Gamma g r_j}$ , where  $r_j$  are a system of left coset representatives for  $\Gamma_{g^{-1}}$  in  $\Gamma$  (thus  $\Gamma = \bigcup \Gamma_{g^{-1}} r_j$ ) and then  $\Gamma g \Gamma = \bigcup \Gamma g r_j$ . Then by the above result (and since  $\Gamma_{g r_j} = \Gamma_g$ ) we obtain that the inverse of  $\widehat{\Gamma g r_j}$  on the set  $(g r_j)^{-1} \Gamma_{g r_j} \alpha_i^{-1} S \cap$

$S = (gr_j)^{-1}\Gamma_g\alpha_i^{-1}S \cap S$  is  $\widehat{\Gamma r_j^{-1}g^{-1}\alpha_i} = \widehat{\Gamma g^{-1}\alpha_i}$  acting on the set  $\alpha_i^{-1}\Gamma_g gr_j S \cap S = \alpha_i^{-1}g\Gamma_{g^{-1}r_j}S \cap S$ . (Here  $\Gamma = \bigcup \Gamma_g\alpha_i = \bigcup \Gamma_{g^{-1}r_j}$ ). Note that the sets  $g^{-1}r_j^{-1}\Gamma_g\alpha_i S \cap S$  are disjoint after  $i$ , while the sets  $\alpha_i^{-1}g\Gamma_{g^{-1}r_j}S \cap S$  are disjoint after  $j$ .

In the following we want to describe the algebra  $\mathcal{B}$  of subsets of  $S$  (and hence of  $F$ ) that are invariated by the transformations  $\widehat{\Gamma}g$  taken on their domains of bijectivity.

Clearly,  $\mathcal{B}$  contains first all sets of the form  $\alpha_i^{-1}\Gamma_g g S \cap S$ , and  $g^{-1}\Gamma_g\alpha_i S \cap S$ , for all  $g \in G$ ,  $\Gamma = \bigcup \Gamma_g s_i$ .

The sets  $g^{-1}\Gamma_g\alpha_i S \cap S$  are easily written in the form

$$(A) \quad \{s \mid gs \in \Gamma_g\alpha_i S \cap S\}$$

while the sets  $\alpha_i^{-1}\Gamma_g g S \cap S = \alpha_i^{-1}g\Gamma_{g^{-1}}S \cap S$  are

$$(B) \quad \{s \in S \mid \alpha_i^{-1}gs \in \Gamma_{g^{-1}}S\}.$$

It is clear that by decomposing  $\Gamma_g$  with respect to smaller normal subgroup  $\Gamma_0 \subseteq \Gamma_g$  as  $\Gamma_g = \bigcup_a \gamma_a \Gamma_0$ , the sets in formula (A) are also of the form

$$\{s \mid gs \in \Gamma_a\alpha_i\Gamma_0 S \cap S\}.$$

**Proposition 5.** *The Borel algebra  $\mathcal{B}$  defined above is invariant under the transformation of the type  $\widehat{\Gamma}g$ ,  $g \in G$ .*

*Proof.* It is sufficient to do this on a domain of bijectivity. Hence for  $g \in G$ , we let  $(\alpha_i)$  be a system of representatives for  $\Gamma \setminus G$ , that is  $\Gamma = \bigcup \Gamma_g\alpha_i$  and consider the restriction of  $\widehat{\Gamma}g$  to  $g^{-1}\Gamma_g\alpha_i F \cap F = \{s \in F \mid gs \text{ belongs to } \Gamma_g\alpha_i F\}$ .

Let  $A_{g_1, \dots, g_n, \Gamma_0} = \{s \in F \mid g_i s \in \Gamma_0 F\}$  be one of the generators of the algebra  $\mathcal{B}$  ( $\Gamma_0$  a subgroup of  $\Gamma$ ).

We will consider the Borel algebra of subsets of  $S$  (or  $F$ ) of sets of the form  $A_{g_1, g_2, \dots, g_n, \Gamma_1, \dots, \Gamma_n}$

$$\{s \in S \mid g_1 s \in \Gamma_1 F, \dots, g_n s \in \Gamma_n F\}$$

where  $g_1, g_2, \dots, g_n \in G$ ,  $\Gamma_1, \dots, \Gamma_n$  are subgroups of  $\Gamma$  of finite index, in the directed subset  $\mathcal{S}$  of subgroups of  $\Gamma$  of the form  $\Gamma_g$ .

It is clear that by dividing  $\Gamma_i$  into cosets with respect to a smaller common subgroup  $\Gamma_0$ , we arrive at the situation where we only work with the

Borel subalgebra of subsets of  $S$  (or of  $F$ ) of subsets of the form

$$A_{g_1, \dots, g_n, \Gamma_0} = \{s \in F \mid g_i s \in \Gamma_0 F\}.$$

If we only want to work with  $g_i$  in a fixed system  $R$  of representatives for  $\Gamma \setminus G$  in  $G$ , then with  $\gamma_j$  a system of representatives for  $\Gamma_0$  in  $\Gamma$  (that is  $\Gamma = \bigcup \gamma_j \Gamma_0$ ),  $j = 1, 2, \dots, [\Gamma : \Gamma_0]$ , then we alternatively take the following generators for the Borel algebra  $\mathcal{B}$ :

$$A_{g_1, \dots, g_n, r_{j_1}, \dots, r_{j_n}, \Gamma_0} = \{s \in F \mid g_i s \in r_{j_i} \Gamma_0 F, i = 1, 2, \dots, n\}.$$

$g_1, \dots, g_n$  run over the system of representatives  $R$ ,  $j_1, \dots, j_n \in \{1, 2, \dots, [\Gamma : \Gamma_0]\}$ .

Since  $\hat{\Gamma}g$  is bijective on

$$(c) \quad \{s \mid gs \text{ belongs to } \Gamma_g \alpha_1 F\}$$

we let  $\Gamma_1$  a smaller normal subgroup (we will determine later how small it has to be taken), and decompose  $\Gamma_g = \bigcup r_j \Gamma_1$ .

So that, the previous set in formula (c) becomes the disjoint union of the sets

$$\{s \mid gs \in r_j \alpha_i \Gamma_1 F\}.$$

We want to determine the image through  $\hat{\Gamma}g$  of the set

$$(***) \quad \{s \mid gs \in r_j \alpha_i \Gamma_1 F\} \cap \{s \mid g_i s \in \Gamma_0 F\}.$$

Note that because of normality of  $\Gamma_1$ , we have that  $r_j \alpha_i \Gamma_1 = \Gamma_1 r_j \alpha_i$ .

Then fix  $f$  are element in the set  $(***)$ . Then  $gf$  is of the form  $\theta_1 r_j \alpha_i f_1$  with  $\theta_1$  in  $\Gamma_1$  and  $f_1 \in F$ . Moreover,  $g_i f \in \Gamma_0 F$ . Then  $\hat{\Gamma}g f = f_1$ , and  $f = g^{-1} \theta_1 r_j \alpha_i f_1$ . The condition that  $g_i f \in \Gamma_0 F$  then translates into  $g_i (g^{-1} \theta_1 r_j \alpha_i) f_1 \in \Gamma_0 F$ , which is the same as

$$f_1 \in \alpha_i^{-1} r_j^{-1} \theta_1^{-1} g g_1^{-1} \Gamma_0 F.$$

We take  $\Gamma_1$  so small that  $\Gamma_1 g g_1^{-1} = g g_1^{-1} \Gamma_2$  for some subgroup  $\Gamma_2$  of  $\Gamma_0$ . Hence the condition on  $f_1$  is that

$$f_1 \in \alpha_i^{-1} r_j^{-1} g g_i^{-1} \Gamma_0 F.$$

We also have to write down the condition that  $f_1$  belongs to the image of  $\{s \mid gs \in r_j \alpha_i \Gamma_1 F\}$  through  $\hat{\Gamma}g$ . But for all  $j$ , we have  $f = g^{-1} \theta r_j \alpha_i f_1$  so  $f_1 = \alpha_i^{-1} r_j^{-1} \theta^{-1} g f$  and hence  $f_1 \in \alpha_i^{-1} r_j^{-1} \Gamma_1 g F \cap F \subseteq \alpha_i^{-1} \Gamma_g g \Gamma \cap F$  is

$$\{s \in F \mid r_j \alpha_i s \in \Gamma_1 g F\} \cap \{g_i g^{-1} r_j \alpha_i s \in \Gamma_0 F\} \quad i = 1, 3, \dots, n.$$

Note that the first set is also  $\{s \in S \mid g^{-1}r_j\alpha_i s \in g\Gamma_1 g^{-1}F\}$  where  $g\Gamma_1 g^{-1} \subseteq \Gamma$  since  $\Gamma_1 \subseteq \Gamma_{g^{-1}}$ .  $\square$

**Observation 6.** The above remark allows to construct a universal groupoid crossed product algebra of  $\{\hat{\Gamma}g\}$  acting on  $B$ , which is universal. If we were able to determine the measure of the sets  $A_{g_1, \dots, g_n, \Gamma_0}$  this would pick all the information from the action, and in the case of a type II Hecke algebra, the measure would induce a trace on the cross product algebra  $(\Gamma \setminus G) \rtimes B$ .

**Theorem 7.** Consider now the case  $G_p = \text{PGL}_2(\mathbb{Z}[\frac{1}{p}])$ ,  $\Gamma = \text{PSL}_2(\mathbb{R})$ , and  $G_{p^n} = \begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}$ . We assume that  $G$  acts freely a.e.  $p \geq 3$ , a prime.

Then the set of piecewise transformations  $\{\hat{\Gamma}g|_{g^{-1}\Gamma_g s_i^g F \cap F} \mid \Gamma g \text{ a system of representatives for } \Gamma\sigma_p\Gamma \text{ cosets, and } \Gamma = \bigcup \Gamma_g s_i^g\}$ , is closed under taking the inverse (this is valid in general if  $\Gamma\sigma\Gamma = \Gamma\sigma^{-1}\Gamma$ ). Moreover, any relation  $\hat{\Gamma}g_1 \hat{\Gamma}g_2 \dots \hat{\Gamma}g_n f = f$ , for some  $f \in F$ , is possible if and only if one of the  $\hat{\Gamma}g_i$  is canceled with its consecutive inverse. In particular, the equivalence relation  $\mathcal{R}_G|F$  is treeable, of cost  $\frac{p+1}{2}$  (its generators and inverses being  $\hat{\Gamma}g$ , with  $\Gamma g \subseteq \Gamma\sigma_p\Gamma$ ).

By Hjorth theorem, there exists a free group factor  $F_{\frac{p+1}{2}}$  acting freely on  $F$ , whose orbits are the equivalence relation in  $\mathcal{R}_G|F$ .

In addition, we can arrange that that generators of  $F_{\frac{p+1}{2}}$  are built only out of the transformations of  $\hat{\Gamma}g$ ,  $\Gamma g \subseteq \Gamma\sigma_p\Gamma$ , and hence the radial elements  $F_{\frac{p+1}{2}}$  (that is  $\chi_n = \text{sum of words in the generators of length } n, n \in \mathbb{N}$ ) have the property that  $\chi_n$  as an operator on  $L^2(F)$  coincides with the Hecke operator  $T_{\sigma_{p^n}}$ .

*Proof.* To prove that treability, recall ([Serre]) that the action of  $\Gamma\sigma_p\Gamma$  on the cosets in  $\Gamma \setminus G_p$  copies exactly the action of the radial algebra on the elements of the free group  $F_{\frac{p+1}{2}}$ . By this we mean that the Cayley tree of  $F_{\frac{p+1}{2}}$  with origin  $e$  is identified with  $\Gamma \setminus G$  (with elements of length  $n$  corresponding to cosets in  $\Gamma\sigma_{p^n}\Gamma$ ). In this way the multivalued action of  $\chi_1 = \sum_{i=1}^{\frac{p+1}{2}} s_i + s_i^{-1}$ , where  $s_i$  are the generators of  $F_{\frac{p+1}{2}}$ , on  $F_{\frac{p+1}{2}}$ , corresponds bijectively to multiplication by  $\Gamma\sigma_p\Gamma$  in the space of cosets. (More precisely, there exists a bijection  $\Psi : F_{\frac{p+1}{2}} \rightarrow \Gamma \setminus G_p$  such that  $\Psi$  preserves length of coset and  $\Psi(\{s_i, s_i^{-1}, i = 1, 2, \dots, \frac{p+1}{2}\}w)$  consists of cosets in  $[\Gamma\sigma_p\Gamma]\Psi(w)$ .)



Thus in any sequence  $\widehat{\Gamma g_1} \widehat{\Gamma g_2} \dots \widehat{\Gamma g_n} f = f$ ,  $f \in F$  with  $\Gamma g_1, \Gamma g_2, \dots, \Gamma g_n$  cosets in  $[\Gamma \sigma_p \Gamma]$ , which will corresponds to some cancellation,

$$\gamma g_1 r_j g_2 r_j \dots g_{n-1} r_j g_n f = f$$

this will be possible if we have successive cancellations of the form  $g_j r_j g_{j+1} \in \Gamma$ , which correspond to multiply in  $\widehat{\Gamma g_j}$  with its inverse.

Thus the equivalence relation is treeable, with generators and inverses, being the transformations  $\widehat{\Gamma g}$  restricted to bijectivity domains.

Since this set is closed under inverses and the total area of the domains is  $p + 1$  it follows that the cost of the relation is  $\frac{p+1}{2}$ . (This is easily extended for  $p = 2$ .)

By Hjorth theorem [Hj], we can find a free group  $F_{\frac{p+1}{2}}$  whose orbits define the relation  $\mathcal{R}_{G_p}|F$ . Since we have that the point images and point preimages of  $\widehat{\Gamma g}$  have cardinality exactly  $[\Gamma : \Gamma_p]$  in the set  $F$ , it follows that by using sets in the boolean algebra generated by bijectivity domains, we can arrange so that the generators  $s_1, s_2, \dots, s_{\frac{p+1}{2}}$  of the group  $F_{\frac{p+1}{2}}$  are built only out of pieces of  $\widehat{\Gamma g}, \Gamma g \subseteq \Gamma \sigma_p \Gamma$ .

But then  $\sum s_i + s_i^{-1}$  is the Hecke operator  $[\Gamma \sigma_p \Gamma]$  acting on  $F$ .  $\square$

We introduce the following definition/corollary of the preceding

**Corollary 8.** *Let  $H_1, H_2$  be two discrete groups. We will say that  $H_1$  is an infinitesimal orbit reduction of  $H_2$ , if there exist an infinite ergodic measure preserving free a.e. action of  $H_2$ , and  $F$  a finite measure subset of  $Y$ , such that if  $\mathcal{R}_{H_2}$  the countable equivalence relation induce on  $Y$  by the orbits of  $H_2$ , then  $\mathcal{R}_{H_2}|F$  is orbit equivalent to an action of  $H_1$ . Thus, we proved that  $F_{\frac{p+1}{2}}$  is infinitesimal orbit equivalent to  $\text{PGL}_2(\mathbb{R}[\frac{1}{p}])$ .*

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